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THE CANONICAL FORMULATION OF CONSERVATIVE ELECTROMAGNETIC FIELDS IN CAVITIES

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This paper develops a theory of frequencies in an electromagnetic field within a resonator loaded with an arbitrary system of current-conducting elements. By means of the canonical substitution of generalized coordinates, the Maxwell equations are brought to a canonical form, formally identical to the equations of motion of a chain of oscillators. Conclusions are made with regard to the Lagrangian function that describes the interaction of an electromagnetic field with the currents it induces in the conductive elements of a passive load.

1 INTRODUCTION

The subject of the present work is the theory of natural oscillations of electromagnetic fields in resonators, in whose cavities is distributed a certain passive load. The traditional elements of such a load in the resonators of linear ion accelerators are a system for stabilization of the electromagnetic field, drift tubes with suspension stems, rods for the regulation of ion energy, and the like. With regard to the effect on the electromagnetic field, load resonators can be classified as resonant or nonresonant. The influence of a nonresonant load shows itself basically in variation of the frequency spectrum of natural oscillations of the electromagnetic field. These variations are proportional to the volume of the element introduced into the resonator, which is not as a rule large. An exception is a resonant condition, under which the frequency of natural oscillations of the field load coincides with one of the natural frequencies of the field oscillations in the resonator. Under these conditions, even an insignificant alteration in the parameters of the load results in a large change of the spectral composition of the electromagnetic field and the values of its natural frequencies. These properties of resonant loads are utilized for regulation of the energy of ions during acceleration and for the stabilization of the electromagnetic field in linear accelerators.[1–4]

In the present study, we obtain a canonical form of the equations of an electromagnetic field within a resonator with a passive load. These equations agree formally with Hamilton's equations for a chain of coupled oscillators.

2 THE LAGRANGIAN OF A CONSERVATIVE ELECTRODYNAMIC SYSTEM

A conservative electrodynamic system is defined as an electromagnetic field in a resonator with ideally conductive walls, within which is arranged a system of perfect conductors. Inasmuch as the geometric form of these conductors and their orientation to relative coordinate axes may be arbitrary, finding a solution of Maxwell's equations that satisfies corresponding boundary conditions is impossible. It therefore becomes justifiable to move away from a field description of the system to a discrete description of the system through an introduction of a corresponding system of canonical variables. The most convenient way of realizing this procedure is to proceed from the Principle of Least Action.

The Lagrangian of a system that does not contain high-energy particles is[5]

$$L = \frac{1}{8\pi}\int_V (E^2 - H^2)\mathrm{d}v, \tag{1}$$

where the integration takes place over the entire volume not occupied by conductors. Equation (1), together with Hamilton's principle,[6] is equivalent to Maxwell's equations. The role of generalized coordinates, which must be varied to find the field equations, is played by components of the four-potential of the electromagnetic field, which is connected with the field vectors **E** and **H** by the equations

$$\mathbf{H} = \nabla \times \mathbf{A}; \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\phi. \tag{2}$$

Before approaching the canonical transformation of the Lagrangian, we will make a few qualitative remarks with regard to the nature of an electromagnetic field in a loaded resonator. For the sake of concreteness, it is assumed here that all the elements of the load are thin rods whose total volume is small compared with the volume of the resonator. The electrical charges and currents induced on the surface of these rods create a so-called "near" field. This field is localized to the region of flow of the currents and, with a great degree of accuracy, may be considered "potential." In the Lagrangian (1), it is convenient to isolate this part of the field in a separate term. The reasoning offered above constitutes a theorem of conceptualization of an arbitrary vector field $\mathbf{V}$ in the form of a superposition of the fields $\mathbf{V}_1$ and $\mathbf{V}_2$ distinguishable from each other in that $\nabla \cdot \mathbf{V}_1 = 0$ but $\nabla \cdot \mathbf{V}_2 \neq 0$. In this, the potential portion of the field is the vector $\mathbf{V}_2$.

The electromagnetic field may then be written in the form of two superposed fields

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \qquad \mathbf{H} = \mathbf{H}_1 + \mathbf{H}_2. \tag{3}$$

The vectors with index 2 in (3) describe the "near," potential, portion of the field. With Eq. (3), the Lagrangian can be written as

$$L = L_1 + L_{12} + L_2, \tag{4}$$

where

$$L_i = \frac{1}{8\pi}\int_V (E_i^2 - H_i^2)\mathrm{d}v; L_{12} = \frac{1}{4\pi}\int_V (\mathbf{E}_1 \cdot \mathbf{E}_2 - \mathbf{H}_1 \cdot \mathbf{H}_2)\mathrm{d}v.$$

In this formulation the first term in the Lagrangian defines the vector component of the electromagnetic field, the last term its quasipotential portion, and the term L_{12} the reaction of the electromagnetic field on the load of the resonator.

Our goal is to present an equation for the electromagnetic field in a loaded resonator in a canonical form, analogous to that of the equations of motion. For this we will sequentially carry out a transformation of all the terms in the Lagrangian function into new canonical variables.

3 THE LAGRANGIAN OF THE VECTOR FIELD

In accordance with Eq. (3), the vector potential of the electromagnetic field with a loaded resonator also appears as the sum of two vector functions

$$\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_1(\mathbf{r}, t) + \mathbf{A}_2(\mathbf{r}_2, t). \tag{5}$$

It should be noted that in describing the potential portion of the field it is useful to utilize as generalized coordinates the magnitudes of the electrical charges induced on the conductive surfaces of the load. In order that this change in variables be canonical, i.e., not changing the appearance of the "equations of motion," it is also necessary to carry out a transformation of the generalized coordinates that describe the vector portion of the field. It appears that such coordinates are the Hertz projections of a vector-potential $\mathbf{\Pi}$ related to the vector potential $\mathbf{A}_1$ by

$$\mathbf{A}_1 = \frac{1}{c}\frac{\partial \mathbf{\Pi}}{\partial t}.$$

The intensities of electric and magnetic fields are expressed in terms of the Hertz potential as

$$\mathbf{E}_1 = \nabla \times \nabla \times \mathbf{\Pi}; \mathbf{H}_1 = \frac{1}{c}\nabla \times \frac{\partial \mathbf{\Pi}}{\partial t}. \tag{6}$$

We substitute $\mathbf{E}_1$ and $\mathbf{H}_1$ into the expression for the function L_1 and use the auxiliary condition $\nabla \cdot \mathbf{\Pi} = 0$. Then after some vector transformations we find

$$L_1 = \frac{1}{8\pi}\int_V \left\{[\nabla \times (\nabla \times \mathbf{\Pi})]^2 + \frac{1}{c^2}\frac{\partial \mathbf{\Pi}}{\partial t} \cdot \nabla^2 \frac{\partial \mathbf{\Pi}}{\partial t}\right\}\mathrm{d}v. \tag{7}$$

In a resonator with reasonably simple geometrical form (for example, a sphere, a cylinder, a cube, a torus, and so forth) the function $\mathbf{\Pi}(\mathbf{r}, t)$ may be expressed as a series of orthogonal eigenfunctions

$$\mathbf{\Pi}(\mathbf{r}, t) = \sum_\lambda q_\lambda(t)\mathbf{\Pi}_\lambda(\mathbf{r}), \tag{8}$$

where $\lambda = (m, n, l)$ is an arbitrary triad of integers.

In accordance with its definition, the vector potential $\mathbf{\Pi}$ has no singularities on the surfaces of the load elements. Therefore the integration in Eq. (7) may be formally extended into the area occupied by the load. An additional factor which emerges from these conditions and which is proportional to the volume of the load should be removed by means of the corresponding correction of natural frequencies. We shall disregard this nonresonant effect in our present study. We shall complete the integration in Eq. (7) for a special case in which a cylindrical resonator has radius R and length h. In the cylindrical volume, the Hertz vector $\mathbf{\Pi}(\mathbf{r}, t)$ may be written as the superposition of two interrelated orthogonal vector fields $\mathbf{\Pi}^{(e)}(\mathbf{r}, t)$ and $\mathbf{\Pi}^{(h)}(\mathbf{r}, t)$, which in an empty resonator correspond to E and H waves. On the surfaces of the resonator there are the boundary conditions

$$\mathbf{\Pi}^{(\alpha)} \cdot \boldsymbol{\tau} = 0; \mathbf{n} \cdot \nabla \times \mathbf{\Pi}^{(\alpha)} = 0 \tag{9}$$

and the supplemental conditions

$$\nabla \cdot \mathbf{\Pi}^{(\alpha)} = 0; \mathbf{e}_z \cdot \nabla \times \mathbf{\Pi}^{(e)} = 0; \mathbf{e}_z \cdot \mathbf{\Pi}^{(h)} = 0, \tag{10}$$

where $\alpha = (e, h)$ and $\boldsymbol{\tau}$, $\mathbf{n}$, $\mathbf{e}_z$ are the unit vectors tangent to the border, normal to the border and along the axis of symmetry. Then we find

$$\Pi^{(e)}_{z,\lambda} = -\frac{1}{k_{z\lambda}}\frac{\partial\psi}{\partial z}; \Pi^{(e)}_{r,\lambda} = \frac{k_{z\lambda}}{k_\lambda^2}\frac{\partial\psi}{\partial r}; \quad \Pi^{(e)}_{\phi,\lambda} = \frac{k_{z\lambda}}{k_\lambda^2}\frac{1}{r}\frac{\partial\psi}{\partial\phi}, \tag{11}$$

where $\psi(r, \phi, z) = q_\lambda^{(\alpha)} J_n(k_\lambda r) \sin k_{z\lambda} z \sin(n\phi + \theta_n)$ and $k_{z\lambda} = m\pi/h$, $k_\lambda = k_{nl}/R$, where k_{nl} is the lth root of the Bessel function $J_n(x)$, and m, n, and l are integers. The projections of the Hertz potential $\mathbf{\Pi}_\lambda^{(n)}$ take the form

$$\Pi_z^{(h)} = \frac{1}{k_\lambda r}\frac{\partial\psi}{\partial\phi}; \Pi_\phi^{(h)} = \frac{1}{k_\lambda}\frac{\partial\psi}{\partial r}. \tag{12}$$

Substituting Eqs. (11) and (12) into the expressions for L_1 and carrying out the integrations, we obtain our final expression for the Lagrangian of the vectorial portion of the electromagnetic field

$$L_1 = \sum_{\lambda,\alpha} \left(\tfrac{1}{2}\omega_\lambda^2 Q_{\lambda,\alpha}^2 - \tfrac{1}{2}\dot{Q}_{\lambda,\alpha}^2\right), \tag{13}$$

in which we have introduced new generalized coordinates $Q_{\lambda,\alpha}$ connected with the amplitudes $q_\lambda^{(\alpha)}$ of the Hertz vector through the equation

$$Q_{\lambda,\alpha} = \left(\frac{V}{16\pi}\right)^{1/2} \omega_\lambda^2 \frac{J_n'(k_\lambda R)}{k_\lambda c^3} q_\lambda^{(\alpha)}, \tag{14}$$

where $V = \pi R^2 h$ is the volume of the resonator and ω is its natural frequency; that is

$$\omega_\lambda = c^2(k_{z\lambda}^2 + k_\lambda^2). \tag{15}$$

The Lagrangian in Eq. (13) constitutes the vector portion of the electromagnetic field in the form of a coupled system of oscillators. In the absence of load, formula (13), together with the Lagrangian equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_1}{\partial \dot{Q}_\lambda}\right) - \frac{\partial L_1}{\partial Q_\lambda} = 0, \tag{16}$$

completely describes the electromagnetic properties of a resonator. In this formulation the generalized coordinates Q_λ define the spectral composition of the electromagnetic field, while the numbers ω_λ are its natural frequencies.[7]

4 THE INTERACTION LAGRANGIAN

Let us compute the part of the Lagrangian function L_{12} that describes the interaction of the electromagnetic field with its induced electrical charges and currents on the load elements. Taking into account the quasi-potential nature of the fields determined by the load elements, we express L_{12} as

$$L_{12} = -\frac{1}{4\pi}\int_V \left(\mathbf{E}_1 \cdot \nabla\phi + \frac{1}{c}\mathbf{H}_2 \cdot \nabla \times \frac{\partial\mathbf{\Pi}}{\partial t}\right)\mathrm{d}v. \tag{17}$$

The volume integral in Eq. (17) can be transformed into a surface integral with the help of the known vector identities[8]

$$\mathbf{E} \cdot \nabla\phi = \nabla \cdot (\phi\mathbf{E}) - \phi\nabla \cdot \mathbf{E}$$
$$(\nabla \times \mathbf{B}) \cdot \mathbf{H}_2 = \nabla \cdot (\mathbf{B} \times \mathbf{H}_2) + \mathbf{B} \cdot \nabla \times \mathbf{H}_2. \tag{18}$$

In using Eqs. (18) in Eq. (17), one needs to put $\nabla \cdot \mathbf{E}_1 = 0$ and, to take into account the condition of quasi-potentiality, $\nabla \cdot \mathbf{H}_2 = 0$. Also taking into account that on the surface of the resonator $y = 0$ and $\mathbf{A}_1 \times \mathbf{H}_2 = 0$, on the basis of the theorem of Gauss–Ostrogradski we find

$$L_{12} = -\frac{1}{4\pi}\int_{S_1}\left[\phi E_1 + \frac{1}{c}\left(\frac{\partial\mathbf{\Pi}}{\partial t} \times \mathbf{H}_2\right)\right] \cdot \mathrm{d}\mathbf{S}, \tag{19}$$

where $\mathrm{d}\mathbf{S}$ is the element of the inner normal to the load surface. The integration in Eq. (19) is carried out assuming that the load of the resonator is created by sufficiently thin conductors whose linear cross-section dimensions are small in comparison with the characteristic length of variation of $\mathbf{E}_1$ and $\mathbf{H}_1$. With such a load, the second term in Eq. (19) becomes dominant. Indeed, introducing a local cylindrical coordinate system (ξ, θ, ρ) with its axis of symmetry ξ directed along the tangent to the axis of the load element, while taking into account that on the surface S_1 that $(\mathbf{E}_1 + \mathbf{E}_2)\cdot\mathbf{e}_\theta = 0$, we find that the first term in Eq. (19) equals

$$L_{12}^{(e)} = \frac{a^2}{4}\sum_k \int_{\Gamma_k} E_{1n}^2\, \mathrm{d}\xi_k, \tag{20}$$

where a is the radius of the loading rod, which we consider constant, $E_{1n} = |\mathbf{E}_1 \times \mathbf{e}_\xi|$, and integrations along ξ are carried out on the contour Γ_k coinciding with the axis-line of the load.

Let us calculate the second term in the expression for Lagrangian interactions. Near the surfaces of the thin conductors, the azimuthal projection of the magnetic-field intensity has a larger magnitude

$$H_{2\theta} \simeq \frac{2}{c\rho}\, i(\xi);\, (\rho \to a), \tag{21}$$

where $i(\xi)$ is the current flowing along the rod. Then we obtain upon integration along θ

$$L_{12}^{(h)} = \frac{1}{c^2}\int_{\Gamma_k} i(\xi)\,\frac{\partial \mathbf{\Pi}}{\partial t}\cdot \mathrm{d}\boldsymbol{\xi}. \tag{22}$$

Comparing the electric $[L_{12}^{(e)}]$ and magnetic $[L_{12}^{(h)}]$ parts of the Lagrangian L_{12}, we can see that in the case of thin conductors $L_{12}^{(h)} \gg L_{12}^{(e)}$. Therefore in all further calculations we shall neglect the term $L_{12}^{(e)}$.

The physical meaning of the interaction Lagrangian in the form (22) is easy to clarify by the example of a nearly closed loop whose dimensions are small compared with the characteristic length of a wave. In this case the current may be considered the same at all points of the contour and $i(\xi) \equiv i$ may be taken out from under the integral sign. Making use of Stokes' theorem, we get

$$L_{12} = \frac{i}{c^2}\int_{f_k} \nabla \times \frac{\partial \mathbf{\Pi}}{\partial t}\cdot \mathrm{d}\mathbf{f}_k = i\,\frac{\Phi_k}{c}, \tag{23}$$

where Φ_k is the flux of the magnetic field $\mathbf{H}_1$ through the surface f_k bounded by the contour Γ_k. In this way, the interaction Lagrangian L_{12} in absolute value constitutes free energy from the linear current i within an external magnetic field $\mathbf{H}_1$.

5 THE LAGRANGIAN OF THE "NEAR" FIELD

According to Eq. (4) and the definition given for the "near" field, the Lagrangian L_2 is the difference between the energy of electrostatic fields with charged conductors and the energy of magnetic fields with currents flowing in the conductors. Generally, the charge distribution $\rho(\xi)$ and the distribution of current $i(\xi)$ are functions of the point ξ located on the axis of the conductor. By taking these functions on the contour Γ_k as a harmonic series, we obtain an infinite sequence of Fourier coefficients subject to the definition in a corresponding system of Lagrangian equations. Such a procedure, however, appears as unnecessarily cumbersome. As long as the length of the conductor does not exceed the characteristic wave length of the field $\mathbf{\Pi}_1$, as generalized coordinates corresponding to the "near" field, we can accept the size of the induced electrical charge on the surface of the conductor

$$q_k = \int_0^{h_k} \rho(\xi)\mathrm{d}\xi, \tag{24}$$

where $\rho(\xi)$ is the linear charge density† and h_k is the length of the conductor. The generalized velocity component corresponding to the coordinate q_k is averaged in terms of the length of the contour current:

$$\dot{q}_k = \frac{1}{h_k}\int i(\xi)\mathrm{d}\xi. \tag{25}$$

In this approximation the integrals in the formula for L_2 can easily be computed and are

$$\frac{1}{8\pi}\int_V E_2^2\,\mathrm{d}v = \frac{1}{2}\sum_{i,k} C_{ik}^{-1}\, q_i q_k \tag{26}$$

$$\frac{1}{8\pi}\int_V H_2^2\,\mathrm{d}v = \frac{1}{2}\sum_{i,k} \frac{L_{ik}}{2}\, \dot{q}_i \dot{q}_k, \tag{27}$$

where the matrix elements C_{ik} and L_{ik} appear as coefficients of mutual capacitance and mutual

† In the case when the load element does not have contact with the surface of the resonator, the value $q_k \equiv 0$ determined by Eq. (24) is in accordance with the law of conservation of electrical charge. Such a situation occurs, for example, in introducing shorted loops, In a load like this, q_k is taken up by the charge induced on the plate capacitor.

inductance of the conductors forming the load of the resonator.[9]

The interaction Lagrangian (L_{12}) in its accepted approximation based on formulas (8), (14), and (22) assumes the form of

$$L_{12} = \sum_{\lambda, k} f_{\lambda k} \dot{q}_k \dot{Q}_\lambda, \tag{28}$$

where

$$f_{\lambda k} = \left(\frac{16\pi}{V}\right)^{1/2} \frac{cK_\lambda}{\omega_\lambda^2 J_n'(k_\lambda R)} \int_V \mathbf{\Pi}_\lambda(\mathbf{r}) \cdot d\boldsymbol{\xi}. \tag{29}$$

Now one can record the final expression for the Lagrangian in a resonator loaded with an arbitrary system of linear conductors

$$L = \frac{1}{2}\sum_{\lambda, \alpha} (\omega_\lambda^2 Q_{\lambda,\alpha}^2 - \dot{Q}_{\lambda,\alpha}^2) - \sum_{\lambda, \alpha, k} f_{\lambda k} \dot{q}_k \dot{Q}_{\lambda,\alpha} + \frac{1}{2}\sum_{i,k}\left(C_{ik}^{-1} q_i q_k \frac{L_{ik}}{c^2} \dot{q}_i \dot{q}_k\right), \tag{30}$$

where i, k label the conductors introduced into the resonators.

Substituting the function L into the system of Lagrangian equations, we can show that on the actual trajectory systems, the magnitude of H is conserved, where

$$H = \sum\left[\frac{1}{2}(\dot{Q}_{\lambda,\alpha}^2 + \omega_\lambda^2 Q_{\lambda,\alpha}^2) + f_{\lambda k} \dot{q}_k \dot{Q}_{\lambda,\alpha} + \frac{1}{2}\left(\frac{L_{ik}}{c^2} \dot{q}_i \dot{q}_k + C_{ik}^{-1} q_i q_k\right)\right], \tag{31}$$

and the summation proceeds over all indices.

It is not difficult to see that the physical meaning of the magnitude of H is the total energy of the electromagnetic field stored in the resonator. Expressing generalized speeds $\dot{q}$ through generalized momenta $p = \partial L/\partial \dot{q}$, we can look at the function H as the Hamiltonian of the system of coupled oscillators.

Thus a rather complex problem of electrodynamics is reduced to a simple problem of classical mechanics, the solution of which does not present any fundamental difficulties. Applying this formalism, we will examine in the next work certain theoretical problems of smooth regulation of the energy of the accelerated ions and that of the stabilization of the electromagnetic field in linear accelerators.

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